

Constituting Atoms of a σ Algebra via Its Generator

Jinshan Zhang*

Department of Mathematical Sciences, Tsinghua University,
Beijing 100084, China

Abstract

In this paper, a very weak sufficient condition for determining atoms by the generator is presented. The condition, though not being a necessary one, is shown to be almost the weakest one in the sense that it can hardly be improved.

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1 Introduction

The atom is an advanced concept in modern probability theory. An intensive application of the properties of atoms plays an important role in one of the proofs of the famous Vatali-Hahn-Saks Theorem [7]. This concept is essential in the development of conditional probability during the recent decades[1, 3, 5, 6]. However when atom is used, the relations between σ algebra and its atoms is mostly assumed to be known. Namely most of the authors consider such relations as conditions in their research. This paper focuses on how to determine such relations, which means to find the structures and the constructions of atoms of a σ algebra. The most relevant result to this paper should be Blackwell theorem [4], which is quite useful and has reduced inclusion between Blackwell σ algebras to comparing their atoms. Hence, determining the atoms of a σ algebra becomes so significant when applying that powerful theorem to solve problems.

The σ algebra itself, however, is usually too large or can not be efficiently obtained by given its generator only. To solve this matter, reducing constituting atoms

*Electronic address: zjs02@mails.tsinghua.edu.cn

from the σ algebra to constituting them from the generator would be a feasible and efficient approach. Therefore, at least some sufficient conditions for the generator on its structure, which can be used to constitute the atoms of the σ algebra generated, should be provided. The condition for the generator proposed in this paper only requires $\kappa(\mathcal{C}) = \sigma(\mathcal{C})$. This condition, though not being a necessary one, is shown to be almost the weakest one in the sense that it can hardly be improved.

The following is a brief road-map of the paper. In section 2, some preliminary knowledge is introduced, which may be used in this paper. We go on in section 3 to prove the main theorem of the paper. Some useful corollaries following the theorem construct the subject of section 4. In section 5, the condition in the main theorem is discussed through specific examples and theoretical analysis. Concluding remarks are proposed.

2 Preliminaries

In this section we review some fundamental concepts and related results that would be utilized in this paper. They are mostly important definitions and theorems in probability theory. In the following, we always let Ω be a set, and \mathcal{C} be a collection (set) of subsets of the set Ω .

Definition 2.1 \mathcal{C} is called a monotone class, if $A_n \in \mathcal{C}, n \geq 1, A_n \uparrow A$ or $A_n \downarrow A \Rightarrow A \in \mathcal{C}$. Further, \mathcal{C} is called a λ class, if \mathcal{C} is a monotone class, $\Omega \in \mathcal{C}$, and $\forall A, B \in \mathcal{C}, B \subset A \Rightarrow A \cap B^c \in \mathcal{C}$.

Definition 2.2 Let (Ω, \mathcal{F}) be a measurable space, where \mathcal{F} is a σ algebra on Ω . For any $\omega \in \Omega$ define $\mathcal{F}_\omega = \{B \in \mathcal{F} | \omega \in B\}$. Then $A(\omega) = \bigcap_{B \in \mathcal{F}_\omega} B$ is called an atom of \mathcal{F} containing ω .

Let $\mathcal{C}_{\cap f} = \{A | A = \bigcap_{i=1}^n A_i, A_i \in \mathcal{C}, i = 1, \dots, n, n \geq 1\}$ be the set closed under finite intersection. Similarly, $\mathcal{C}_{\cup f}, \mathcal{C}_{\sum f}, \mathcal{C}_\delta, \mathcal{C}_\sigma, \mathcal{C}_{\sum \sigma}$ denote the sets closed under finite union, finite disjoint union, countable intersection, countable union, countable disjoint union respectively. $\mathcal{C}_{\sigma\delta} = (\mathcal{C}_\sigma)_\delta$. $\sigma(\mathcal{C}), \lambda(\mathcal{C}), m(\mathcal{C})$ denotes the minimum σ

algebra, λ class, and monotone class containing \mathcal{C} respectively and \mathcal{C} is the generator of them.

3 Main theorem

Let \mathcal{C} be a collection of sets on Ω and $\mathcal{F} = \sigma(\mathcal{C})$. For any $\omega \in \Omega$, define $\mathcal{C}_\omega = \{B \in \mathcal{C} \mid \omega \in B\}$ and $A(\omega)$ the atom containing ω of \mathcal{F} . Question is: under what condition on \mathcal{C} , there would be

$$A(\omega) = \bigcap_{B \in \mathcal{C}_\omega} B.$$

Yan [7] shows that this is true, if \mathcal{C} is an algebra. A much weaker condition on \mathcal{C} is proposed in this section, which is the main result of this paper. In order to show the main result of this paper, we introduce the following concept, which is created in this paper to show how good our condition is.

Definition 3.1 \mathcal{C} is called a κ class, if it is closed under countable intersection and countable union.

Denote by $\kappa(\mathcal{C})$ the minimum κ class containing \mathcal{C} , and \mathcal{C} is called generator of $\kappa(\mathcal{C})$. To complete the proof of the main result, we need the following lemma.

Lemma 3.1 Let \mathcal{C} be a collection of sets on Ω , $\mathcal{F} = \sigma(\mathcal{C})$. $\forall \omega \in \Omega$, define $\mathcal{C}_\omega = \{B \in \mathcal{C} \mid \omega \in B\}$ and $A_{\mathcal{C}}(\omega) = \bigcap_{B \in \mathcal{C}_\omega} B$. Let $\mathcal{G} = \{B \in \mathcal{F} \mid \omega \notin B, \text{ or } \omega \in B \text{ and } A_{\mathcal{C}}(\omega) \subset B\} = \{B \in \mathcal{F} \mid \omega \notin B\} \cup \{B \in \mathcal{F} \mid \omega \in B, A_{\mathcal{C}}(\omega) \subset B\}$.

Then \mathcal{G} satisfies the following three properties:

- 1). $\mathcal{C} \subset \mathcal{G}$, $\mathcal{C}_\sigma \subset \mathcal{G}$.
- 2). \mathcal{G} is closed under the operation of countable union and countable intersection.
- 3). \mathcal{G} is a κ class. In particular, it is a monotone class.

Proof: Let $\mathcal{C}_{\sigma\omega} = \{B \in \mathcal{C}_\sigma \mid \omega \in B\}$ and $A_{\mathcal{C}_\sigma}(\omega) = \bigcap_{B \in \mathcal{C}_{\sigma\omega}} B$. Claim $\forall \omega \in \Omega$, $A_{\mathcal{C}}(\omega) = A_{\mathcal{C}_\sigma}(\omega)$. First, $\mathcal{C}_\omega \subset \mathcal{C}_{\sigma\omega}$, then $A_{\mathcal{C}_\sigma}(\omega) = \bigcap_{B \in \mathcal{C}_{\sigma\omega}} B \subset \bigcap_{B \in \mathcal{C}_\omega} B = A_{\mathcal{C}}(\omega)$. Consider the definition of \mathcal{C}_σ and $\mathcal{C}_{\sigma\omega}$, $\forall B \in \mathcal{C}_{\sigma\omega}$, $\exists \{A_n\}_{n=1}^\infty \subset \mathcal{C}$ such that $B = \bigcup_{n=1}^\infty A_n$. Hence, there exists N such that $\omega \in A_N$, then there are $A_{\mathcal{C}}(\omega) \subset A_N \subset B$ and $A_{\mathcal{C}}(\omega) \subset \bigcap_{B \in \mathcal{C}_{\sigma\omega}} B = A_{\mathcal{C}_\sigma}(\omega)$. Hence $A_{\mathcal{C}}(\omega) = A_{\mathcal{C}_\sigma}(\omega)$. Now, let's prove the lemma.

For the property 1. $\forall B \in \mathcal{C}$, if $\omega \notin B$ then $B \in \mathcal{G}$; Otherwise, if $\omega \in B$, since $A_{\mathcal{C}}(\omega) = \bigcap_{B \in \mathcal{C}_\omega} B$, we have $A_{\mathcal{C}}(\omega) \subset B$, then $B \in \mathcal{G}$. Hence $\mathcal{C} \subset \mathcal{G}$. From the claim, we know $\mathcal{G} = \{B \in \mathcal{F} \mid \omega \notin B, \text{ or } \omega \in B \text{ and } A_{\mathcal{C}_\sigma}(\omega) \subset B\} = \{B \in \mathcal{F} \mid \omega \notin B\} \cup \{B \in \mathcal{F} \mid \omega \in B, A_{\mathcal{C}_\sigma}(\omega) \subset B\}$. Thus, similarly, $\mathcal{C}_\sigma \subset \mathcal{G}$.

For the property 2. Suppose $\{A_n\}_{n=1}^\infty \subset \mathcal{G}$.

- (i). If $\forall n, \omega \notin A_n$, then $\omega \notin \bigcup_{n=1}^\infty A_n$. Hence $\bigcup_{n=1}^\infty A_n \in \mathcal{G}$.
- (ii). If $\exists n$ such that $\omega \in A_n$ then $A_{\mathcal{C}}(\omega) \subset A_n \subset \bigcup_{n=1}^\infty A_n$. Obviously, $\omega \in \bigcup_{n=1}^\infty A_n$.

Hence $\bigcup_{n=1}^\infty A_n \in \mathcal{G}$

Considering (i) and (ii), \mathcal{G} is closed under countable union.

- (iii). If $\exists n$ such that $\omega \notin A_n$, then $\omega \notin \bigcap_{n=1}^\infty A_n$. Thus $\bigcap_{n=1}^\infty A_n \in \mathcal{G}$.
 - (iv). If $\forall n, \omega \in A_n$, then $\omega \in \bigcap_{n=1}^\infty A_n$. Since $A_{\mathcal{C}}(\omega) \subset A_n(\forall n)$, $A_{\mathcal{C}}(\omega) \subset \bigcap_{n=1}^\infty A_n$.
- Hence $\bigcap_{n=1}^\infty A_n \in \mathcal{G}$.

Considering (iii) and (iv), \mathcal{G} is closed under countable intersection.

For property 3. From property 2, we know \mathcal{G} is a κ class. In particular, if $A_n \uparrow A$ then $A = \bigcup_{n=1}^\infty A_n$, and if $A_n \downarrow A$, then $A = \bigcap_{n=1}^\infty A_n$. Hence \mathcal{G} is a monotone class. \square

Using Lemma 3.1, now we prove the main result of this paper.

Theorem 3.1 Let \mathcal{C} be a collection of sets on Ω , $\mathcal{F} = \sigma(\mathcal{C})$ and $A_{\mathcal{F}}(\omega)$ the atom of \mathcal{F} containing ω . $\forall \omega \in \Omega$, define $\mathcal{C}_\omega = \{B \in \mathcal{C} \mid \omega \in B\}$ and $A_{\mathcal{C}}(\omega) = \bigcap_{B \in \mathcal{C}_\omega} B$. If the generator \mathcal{C} satisfies the property that $\forall A \in \mathcal{C} \Rightarrow A^c \in \kappa(\mathcal{C})$, then

$$A_{\mathcal{F}}(\omega) = A_{\mathcal{C}}(\omega).$$

Proof: $\forall \omega$, let $\mathcal{G}_1 = \{B \in \mathcal{F} \mid \omega \notin B, \text{ or } \omega \in B \text{ and } A_{\mathcal{C}}(\omega) \subset B\}$ and $\mathcal{G}_2 = \{A \in \mathcal{G}_1 \mid A^c \in \mathcal{G}_1\}$. Then \mathcal{G}_2 satisfies the following properties.

- (a). $\forall A \in \mathcal{C}$, $A^c \in \kappa(\mathcal{C})$, by the property 1 and 3 of \mathcal{G}_1 in Lemma 3.1, we know $\kappa(\mathcal{C}) \subset \mathcal{G}_1$, then $A^c \in \mathcal{G}_1$. Hence $\mathcal{C} \subset \mathcal{G}_2$.

- (b). Since \mathcal{G}_1 is a monotone class, it's easy to check \mathcal{G}_2 is a monotone class.

- (c). Now let's check \mathcal{G}_2 is an algebra.

- (i). $\forall A \in \mathcal{G}_2$, then $A \in \mathcal{G}_1$, $A^c \in \mathcal{G}_1$, $(A^c)^c \in \mathcal{G}_1$, hence $A^c \in \mathcal{G}_2$.

- (ii). $\forall A, B \in \mathcal{G}_2$, then $A, A^c \in \mathcal{G}_1$ and $B, B^c \in \mathcal{G}_1$. Consider the property 2 of \mathcal{G}_1 ,

we know $A \cap B \in \mathcal{G}_1$, $A^c \cup B^c \in \mathcal{G}_1$, then $(A \cap B)^c \in \mathcal{G}_1$. Hence $A \cap B \in \mathcal{G}_2$.

Considering (i) and (ii), we show \mathcal{G}_2 is an algebra. Now from (a), (b) and (c), \mathcal{G}_2 is a monotone class and algebra containing \mathcal{C} . By Monotone Class Theorem, $\mathcal{F} = \sigma(\mathcal{C}) \subset \mathcal{G}_2$. Then $\mathcal{G}_2 = \mathcal{F}$. $\mathcal{G}_2 \subset \mathcal{G}_1 \subset \mathcal{F}$, then $\mathcal{G}_1 = \mathcal{F}$. Noting that $\mathcal{G}_1/\mathcal{F}_\omega = \{B \in \mathcal{F} \mid \omega \notin B\}$ (recall $\mathcal{F}_\omega = \{B \in \mathcal{F} \mid \omega \in B\}$), then $\forall B \in \mathcal{F}_\omega$ $A_{\mathcal{C}}(\omega) \subset B$. Hence $A_{\mathcal{C}}(\omega) \subset A_{\mathcal{F}}(\omega)$. since $\mathcal{C}_\omega \subset \mathcal{F}_\omega$, $A_{\mathcal{F}}(\omega) \subset A_{\mathcal{C}}(\omega)$. Thus the result of this theorem follows. \square

4 Corollaries

In this section useful corollaries following the main theorem is presented.

Corollary 4.1 If \mathcal{C} is a semi-algebra on Ω , and $\mathcal{F} = \sigma(\mathcal{C})$. Then $\forall \omega \in \Omega$,

$$A_{\mathcal{C}}(\omega) = A_{\mathcal{F}}(\omega).$$

Proof: For any $A \in \mathcal{C}$, one has

$$A^c = \Omega/A \in \mathcal{C}_{\Sigma f} \subset \mathcal{C}_\sigma \subset \kappa(\mathcal{C}_\sigma) = \kappa(\mathcal{C}).$$

Hence the result follows. \square

Corollary 4.2 If \mathcal{C} is a semi-ring, $\Omega \in \mathcal{C}_\sigma$, $\mathcal{F} = \sigma(\mathcal{C})$. Then $\forall \omega \in \Omega$,

$$A_{\mathcal{C}}(\omega) = A_{\mathcal{F}}(\omega).$$

Proof: There exists a sequence $A_n \in \mathcal{C}$ such that $\Omega = \bigcup_{n=1}^{\infty} A_n$. Then

$$A^c = \bigcup_{n=1}^{\infty} (A_n/A),$$

by noting $A_n/A \in \mathcal{C}_{\Sigma f} \subset \mathcal{C}_\sigma$. Hence $A^c \in \mathcal{C}_\sigma \subset \kappa(\mathcal{C})$. \square

Corollary 4.3 If $\kappa(\mathcal{C}) = \sigma(\mathcal{C})$, then $\forall \omega \in \Omega$, $A_{\mathcal{C}}(\omega) = A_{\mathcal{F}}(\omega)$.

Proof: We show the equivalence between $\kappa(\mathcal{C}) = \sigma(\mathcal{C})$ and $\forall A \in \mathcal{C} \Rightarrow A^c \in \kappa(\mathcal{C})$.

If $\kappa(\mathcal{C}) = \sigma(\mathcal{C})$, obviously, there are $\forall A \in \mathcal{C} \Rightarrow A^c \in \kappa(\mathcal{C})$. For the inverse direction, the collection of set $\mathcal{G} = \{B \in \kappa(\mathcal{C}) \mid B^c \in \kappa(\mathcal{C})\}$, which is closed under countable

intersection, countable union and complement, contains \mathcal{C} . Hence, \mathcal{G} is a σ algebra and $\mathcal{G} = \kappa(\mathcal{C}) = \sigma(\mathcal{C})$. \square .

In the following corollaries we suppose that \mathcal{F} is separable(\mathcal{F} can be generated by a countable subset), so they can be directly applied to the comparison among atoms in Blackwell space[4].

Corollary 4.4 Suppose \mathcal{C} is a countable semi-ring and $\mathcal{F} = \sigma(\mathcal{C})$ (Obviously, \mathcal{F} is separable). Then $\forall \omega \in \Omega$, $A_{\mathcal{C}}(\omega) = A_{\mathcal{F}}(\omega)$ if and only if $\Omega \in \mathcal{C}_{\sigma}$.

Proof: From Corollary 4.2, we know we only have to check if $\forall \omega \in \Omega$ $A_{\mathcal{C}}(\omega) = A_{\mathcal{F}}(\omega) \Rightarrow \Omega \in \mathcal{C}_{\sigma}$. $\forall \omega \in \Omega$ $A_{\mathcal{C}}(\omega) = A_{\mathcal{F}}(\omega)$, then $\exists B \in \mathcal{C}$ such that $\omega \in B$. Hence, $\Omega = \bigcup_{B \in \mathcal{C}} B$. Note \mathcal{C} is countable, then $\Omega = \bigcup_{B \in \mathcal{C}} B \in \mathcal{C}_{\sigma}$. \square

Corollary 4.5 Let \mathcal{C} be a collection of sets on Ω , $\mathcal{F} = \sigma(\mathcal{C})$. If \mathcal{F} has countable atoms and \mathcal{C} is countable. Then $\forall \omega \in \Omega$, $A_{\mathcal{F}}(\omega) = A_{\mathcal{C}}(\omega)$ if and only if $\mathcal{F} = \kappa(\mathcal{C})$.

Proof: If $\forall \omega \in \Omega$, $A_{\mathcal{F}}(\omega) = A_{\mathcal{C}}(\omega)$. $\forall A \in \mathcal{C}$, $A^c = \bigcup_{\omega \in A^c} A_{\mathcal{F}}(\omega) = \bigcup_{\omega \in A^c} A_{\mathcal{C}}(\omega)$. Since \mathcal{C} is countable, $A_{\mathcal{C}}(\omega) = \bigcap_{B \in \mathcal{C}_{\omega}} B \in \kappa(\mathcal{C})$. Since the atoms of \mathcal{F} is countable, $\bigcup_{\omega \in A^c} A_{\mathcal{C}}(\omega) = \bigcup_{\omega \in A^c} \bigcap_{B \in \mathcal{C}_{\omega}} B \in \kappa(\mathcal{C})$, indicating $A^c \in \kappa(\mathcal{C})$. From the proof of Corollary 4.3, $A^c \in \kappa(\mathcal{C})$ ($\forall A \in \mathcal{C}$) implies $\mathcal{F} = \kappa(\mathcal{C})$. The inverse that $\mathcal{F} = \kappa(\mathcal{C})$ implies $\forall \omega \in \Omega$ $A_{\mathcal{F}}(\omega) = A_{\mathcal{C}}(\omega)$ is trivial if we note that $\mathcal{F} = \kappa(\mathcal{C})$ implies $\forall A \in \mathcal{C}$, $A^c \in \kappa(\mathcal{C})$. \square

In Corollary 4.1 and 4.2, we do not really use the property of semi-ring or semi-algebra, which is closed under finite intersection. Besides the condition $A \cap B^c \in \mathcal{C}_{\Sigma f}$ can be replaced by $A \cap B^c \in \mathcal{C}_{\sigma\delta}$.

5 Discussion and conclusion

First consider the following two examples.

Example 5.1 Let $\Omega = R$, $\mathcal{C} = \{x | x \in R\}$ and $\mathcal{F} = \sigma(\mathcal{C})$. Obviously, $\forall x \in R$ $A_{\mathcal{C}}(x) = x = A_{\mathcal{F}}(x)$, and \mathcal{F} is Hausdoff(the atoms of \mathcal{F} are the points of Ω). It's

easy to check $\kappa(\mathcal{C}) \subset \{A \subset R \mid A \text{ is countable}\}$. However, $R/\{0\} \in \mathcal{F}$ is not in $\kappa(\mathcal{C})$. This shows our condition is not a necessary one.

Example 5.2 Let $\Omega = [0, 1]$, $\mathcal{C} = \{[a, b) \subset [0, 1] \mid a < b\} \cup \{\emptyset\}$ and $\mathcal{F} = \sigma(\mathcal{C})$. \mathcal{C} is a semi-ring on $[0, 1]$. $A_{\mathcal{C}}(1) = \emptyset$, while $A_{\mathcal{F}}(1) = \{1\}$. This shows the condition that the generator is a semi-ring is not sufficient for $A_{\mathcal{C}}(\omega) = A_{\mathcal{F}}(\omega) (\forall \omega \in \Omega)$.

The examples show that our condition may not be the best one but almost necessary. Comparing our condition with semi-ring (see Corollary 4.2), we only add $\Omega \in \mathcal{C}_{\sigma}$ to obtain the desired result, and the condition of Corollary 4.2 is stronger than that in our main theorem. Therefore our condition has already been a very weak one. On the other hand since $m(\mathcal{C}) \subset m(\mathcal{C}_{\sigma}) \subset \kappa(\mathcal{C}_{\sigma}) = \kappa(\mathcal{C}) \subset \sigma(\mathcal{C})$ and $m(\mathcal{C}_{\sigma}) \subset \lambda(\mathcal{C}_{\sigma}) \subset \sigma(\mathcal{C})$. The trivial case $A^c \in \sigma(\mathcal{C})$ contributes nothing if letting it replace $A^c \in \kappa(\mathcal{C})$ since it is impossible to conclude $A_{\mathcal{C}}(\omega) = A_{\mathcal{F}}(\omega) (\forall \omega \in \Omega)$ without any restriction on \mathcal{C} (Example 5.1 can be viewed as a special counterexample). From relations among $m(\mathcal{C})$, $m(\mathcal{C}_{\sigma})$, $\kappa(\mathcal{C}_{\sigma})$, $\kappa(\mathcal{C})$, $\lambda(\mathcal{C}_{\sigma})$, $\sigma(\mathcal{C})$, we know $\kappa(\mathcal{C})$ is already a very large set and nearly as large as $\lambda(\mathcal{C}_{\sigma})$. Finally, reviewing the proof of the theorem, one can find the key of the proof lies in the property of \mathcal{G} in Lemma 3.1. Generally, \mathcal{G} is at most a κ class and could not be a λ class. Hence the improvement of our condition from the theoretic perspective is almost impossible.

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